Finishing last class

Idea Understand diff'l forms as fns on spaces of maps.

Q1 What is their derivative?

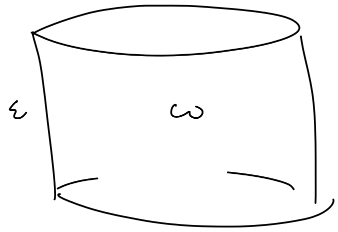
$$\mathcal{L}_X\omega = \iota_X d\omega + d\iota_X\omega$$

- Check for fns
- Check for 1-forms (last time)
- Both are derivations $\rightsquigarrow$ induction

$$\mathcal{L}_X(\omega\wedge\eta) = (\mathcal{L}_X\omega)\wedge\eta + \omega\wedge(\mathcal{L}_X\eta)$$

Interpretation

- Apply on $M\times(0,\varepsilon)$ ← mfld w/ corners



$$\frac{d}{dt}\int_{M\times t}\omega = \int_{M\times t}\mathcal{L}_{\frac{\partial}{\partial t}}\omega$$

$$= \int (d\iota_t + \iota_t d)\omega$$

$$= \int_{\partial M}\iota_t\omega + \int_M \dot{\omega}$$

In particular, if $\partial M = 0$, and $d\omega = 0$, then $\int_{M\times t}\omega$ doesn't depend on t

Cohomology

Dfn θ is · closed if $d\theta = 0$
· exact if $\theta = d\eta$

Lem Closed forms are a sub-algebra and exact forms are an ideal;

Defn $H^i_{dR}(M) =$ Closed/Exact is an algebra w/ $\wedge$
(Hard to prove in simplicial cohomology)

eg for S^1 $H^* = \mathbb{R}[a]/_{a^2=0}$

$a = [d\theta]$

Defn Homotopy

Θ as fn on closed cpct mflds $\to M$
loc. const

M cpct k-mfd-w-∂ S n-mfld, ω k-form

$$\begin{array}{ccc} \mathrm{Map}(M,S) & \xrightarrow[\int \varphi^*\omega]{} & \mathbb{R} \\ \Big\downarrow \text{restriction} & & \\ \mathrm{Map}(\partial M, S) & & \end{array}$$

tangent vector to $\mathrm{Map}(M,S) \sim X$ section of $\varphi^* TS$

variation
$\underbrace{M \times I}_{\text{m-w-c}} \longrightarrow S$

Classification on cobordism
- closed
- Null-cobordant iff all Stiefel-Whitney #'s zero
- Euler characteristic obstruction

Big picture

#1 Language of manifolds   Finished

#2 Riemannian geometry of curves and surfaces in $\mathbb{R}^3$ | w/ induced metric

#3 Use language to generalize (#2) to higher dim'l worlds

Curves in $\mathbb{R}^2$, Curves in $\mathbb{R}^3$, Surfaces in $\mathbb{R}^3$

$C \subseteq \mathbb{R}^n$ viewed as Riemannian mfld.)

curves	intrinsic	extrinsic
Local theory		
Global theory		

Surfaces		

Prop 1 C is locally isometric to $(\mathbb{R}, dx^2)$

Pf Suppose we already have a local chart φ, not necessarily isometric. Let

$$s(t) = \int_{t_0}^{t} |\varphi'(t)| \, dt$$

Since α is regular, s is smooth and $\frac{ds}{dt} > 0$. Therefore, it has an inverse $t(s)$, and

$$\left|\frac{d\varphi}{ds}\right| = \left|\frac{d\varphi}{dt}\right| \left|\frac{dt}{ds}\right| = \frac{|\varphi'(t)|}{s'(t)} = 1 \qquad \lrcorner$$

Rmk φ unique up to translation and reflection.

(This rmk is the beginning of a proof of:

Thm Every connected 1-mfld is diffeomorphic to $\mathbb{R}$ or S^1.

One proof: ① WLOG, metric is complete (dilate on exhaustion)

② If oriented, flow of $\frac{\partial}{\partial x}$: $\Phi : M \times \mathbb{R} \to M$ gives map $\Phi(p, \cdot) : \mathbb{R} \to M$ show image open & closed

③ If not oriented, apply to or. covering (still connected) $\longrightarrow M = \mathbb{R}/\mathbb{Z}_2$ or $S^1/\mathbb{Z}_2$ ← isometries; but must have fixed pt $\to \subset$.

Local extrinsic theory of arclength-parametrized curves:

Q: Classify curves locally up to rigid motion of $\mathbb{R}^n$.

A: measured by turning (instructions for driving)

<u>Canonical frame</u> for $\mathbb{R}^2$

$$\varphi : (a,b) \longrightarrow \mathbb{R}^2 \qquad |\varphi'| = 1$$

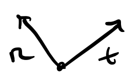
$$t(s) = \varphi'(s)$$
$$n(s) = J\,\varphi'(s)$$

$$J\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

<u>Dfn</u> If $\alpha : I \to \mathbb{R}^2$ param by arclength, let

$t(s) = \alpha'(s)$ the tangent

$n(s) = t(s)^{\perp}$ the normal



Then $\kappa(s) = t'(s) \cdot n(s)$
$= \alpha''(s) \cdot n(s)$
is called the <u>signed curvature</u> of α.

Claim: κ is the derivative of the angle of t:

Let $\theta(s): I \to \mathbb{R}$ be such that $t(s) = \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \end{bmatrix}$ (not unique)

then $t' = \begin{bmatrix} -\sin\theta \, \theta' \\ \cos\theta \, \theta' \end{bmatrix}$, $n = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ $\Rightarrow$ $\kappa = \theta'$

If you are driving the curve at unit speed, κ is the angle of the steering wheel.

Ex: If $\alpha(s) = \begin{bmatrix} r\cos\frac{s}{r} \\ r\sin\frac{s}{r} \end{bmatrix}$ is an arclength param. of a circle of radius r, then $t = \begin{bmatrix} -\sin\frac{s}{r} \\ \cos\frac{s}{r} \end{bmatrix}$, $n = \begin{bmatrix} -\cos\frac{s}{r} \\ -\sin\frac{s}{r} \end{bmatrix}$, $\kappa = \frac{1}{r}$.

Thm

Fundamental theorem of local theory of plane curves:

- For any function $\kappa(s)$ on an interval I, there is an arclength-parametrized curve $\alpha: I \to \mathbb{R}^2$ with curvature $\kappa(s)$
- $\kappa(s)$ determines $\alpha(s)$ uniquely up to rigid motions

$$\theta' = \kappa \qquad\qquad \alpha' = t$$

$$\underset{\text{up to rotation}}{\kappa(s) \xrightarrow{\int} \theta(s)} \xrightarrow{t = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}} \underset{\text{up to translation}}{t(s) \xrightarrow{\int} \alpha(s)}$$

Global theory of plane curves:

Dfn winding number for $S^1 \subseteq \mathbb{R}^2$.

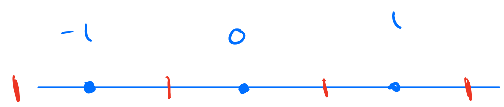
Thm (turning tangents) The winding number of a regular closed curve is ± 1.

Equivalently, an injective regular parametrized closed curve.

Also called a simple closed curve.

Rmk: switching direction negates winding number.

Pf (Turning tangents)



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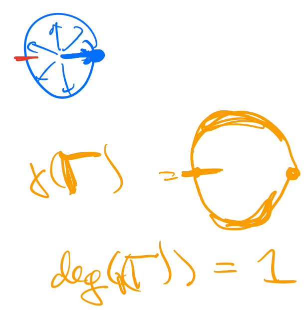
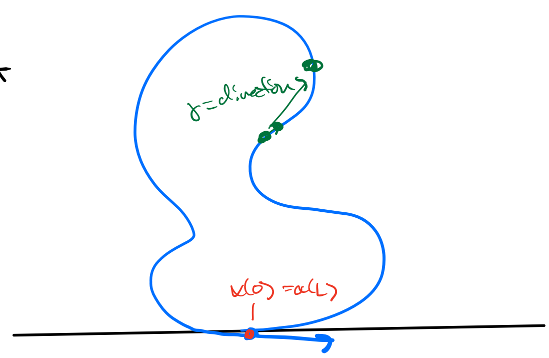
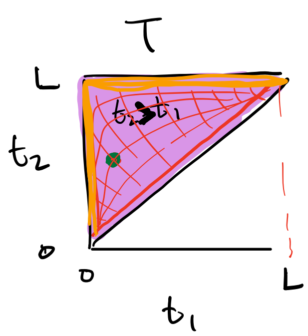


WLOG, base point $\alpha([0,L])$





$$\gamma(t_1, t_2) = \frac{\alpha(t_2) - \alpha(t_1)}{t_2 - t_1}$$

$$\gamma : T \to S^1$$

① $\gamma(t_1, t_1 + \varepsilon) \sim \alpha'(t_1)$

$\Rightarrow \gamma$(red curve) has degree = winding #

(in fact, γ extends continuously to ∂T)

② γ(orange curve) has degree 2π

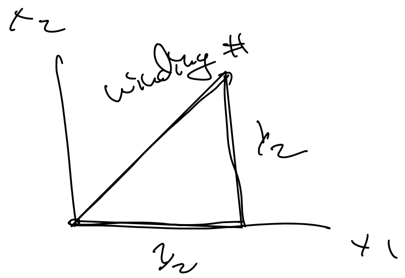
③ homotopic maps have the same degree

same degree. ⌟

Pf $\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}^2$

$\gamma : \{0 \le x_1 \le x_2 \le 1\} \longrightarrow \mathbb{R}^2$

↑ wfd w/ corners



$$\gamma(t_1, t_2) = \begin{cases} \dfrac{\alpha(t_1) - \alpha(t_2)}{t_1 - t_2} & t_1 \neq t_2 \\ \alpha'(t) & t_1 = t_2 = t. \end{cases}$$

$\boxed{\underline{\text{Note}} \text{ degree} = \int \rho^{+} d\theta}$